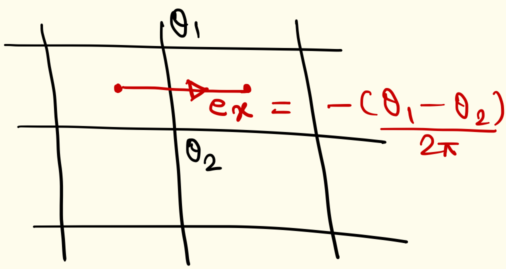
Hamiltonian Formulation of O(2) Duality

Formulating the above duality in terms of a 2+1-d quantum Hamiltonian is illuminating. As you will show in pset-1, the 3D O(2) model corresponds to the following quantum Hamiltonian in 2D:

$$H = -t\sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) + U\sum_i (n_i - \bar{n})^2 \quad \text{where}$$

$\bar{n}$ is an integer (we can set $\bar{n} = 0$).

Introduce bond variables e_{ij} on the dual lattice as



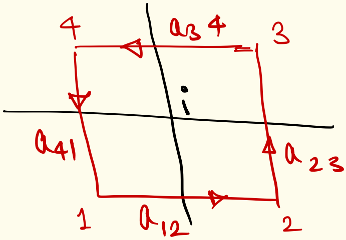
$$\vec{e} = \frac{\hat{z} \times \vec{\nabla}\theta}{2\pi}$$

i.e. $e_x = -\dfrac{\nabla_y \theta}{2\pi}$, $\quad e_y = \dfrac{\nabla_x \theta}{2\pi}$

One also introduces fields conjugate to e_x, e_y:

$$[e_x, a_x] = i = [e_y, a_y] . \quad [a_x, a_y] = [e_x, e_y] = 0.$$

Changing θ_i changes e_{12}, e_{23}, e_{34}, e_{41}.



Since n_i is conjugate to θ_i and e_{ij} to a_{ij}

$$\Rightarrow \quad n_i \propto a_{12} + a_{23} + a_{34} + a_{41}.$$

(the sign follows from $\hat{e} \sim \hat{z} \times \vec{\nabla}\theta$). Putting in all the factors, $n_i = \frac{1}{2\pi}[a_{12} + a_{23} + a_{34} + a_{41}]$

$$\Rightarrow \quad n = \frac{1}{2\pi}(\nabla \times a)$$

Thus, the Hamiltonian is

$$H = -t \sum_{\langle ij \rangle} \cos(2\pi e_{ij}) + U \sum \left(\frac{\nabla \times a}{2\pi}\right)^2$$

One might naively think that $\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot \frac{[\hat{z} \times \nabla\theta]}{2\pi} = 0$

but since θ is a compact variable, it allows for vortices $\Rightarrow \vec{\nabla} \cdot \vec{E} = -\hat{z} \cdot \frac{[\vec{\nabla} \times \vec{\nabla}\theta]}{2\pi}$

$$= n_v = \text{vortex density}.$$

Crucially, e_{ij} is <u>not</u> an integer, it's just a compact variable between 0 to 2π.

On the other hand, $a_{ij}/2\pi$ has integer eigenvalues since it's proportional to a sum of n_i's.

The above theory already looks like a guage theory but it's not yet formulated in terms of vortices. To do that, let's relax the compactness of e (which relaxes the integer-valued ness of $\frac{a}{2\pi}$):

expand $\cos(2\pi e_{ij}) \rightarrow$

$$H \approx \frac{t}{2}\sum(2\pi e_{ij})^2 + U\sum\left(\frac{\nabla\times a}{2\pi}\right)^2 \text{“}-\sum t_v \cos(a)\text{”}$$

with $\nabla\cdot e = n_v$, and t_v enforces the integer valued-ness of $\frac{a}{2\pi}$ softly. However $\cos(a)$ is not guage invariant. Separate a into longitudinal and transverse components: $a = a_\perp + a_\parallel$, where $\nabla\times a_\parallel = 0$, $\nabla\cdot a_\perp = 0$. Write $a_\parallel = -\nabla\theta_v$

$$\Rightarrow H \approx \frac{t}{2}\sum(2\pi e_{ij})^2 + U\sum\frac{(\nabla\times a_\perp)^2}{2\pi}$$

$$- t_v\cos(\nabla\theta_v - a_\perp)$$

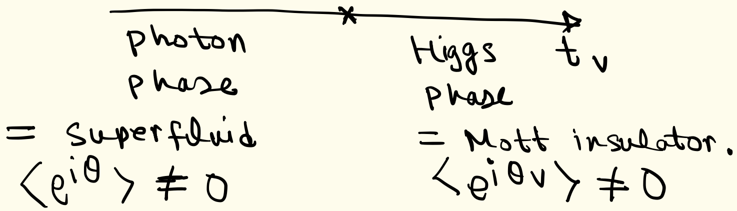
θ_v is the phase of the matter-field (= vortex).

The guage invariance corresponds to:

$$\theta_v \rightarrow \theta_v + \alpha$$
$$a \rightarrow a + \nabla\alpha$$
$$e^{i\alpha(\nabla\cdot E - n_v)} = 1$$

Therefore, this is a theory of point-like particles (= vortices) coupled to a non-compact U(1) guage field.

<u>Phase-diagram:</u>



In the original variables (i.e. θ, n), the superfluid phase has a local order-parameter $\langle e^{i\theta_i} \rangle$. In the dual Hamiltonian however, the photon phase doesn't have a local order parameter. The Higgs phase doesn't break any sym, so it's 'trivial' paramagnet.

The above Hamiltonian also makes it transparent the consequence of condensing pair of vortices without condensing single vortex.

i.e. $\langle e^{2i\theta_v} \rangle \neq 0$, $\langle e^{i\theta_v} \rangle = 0$.

The Higgs term is $\cos(2\theta - 2a_\perp)$

$\sim \cos(2a_\perp)$.

$\Rightarrow$ $U(1)$ guage field gest Higgsed down to $\mathbb{Z}_2$.